MONOIDAL UNIQUENESS OF STABLE HOMOTOPY THEORY

BROOKE SHIPLEY

Abstract. We show that the monoidal product on the stable homotopy category of spectra is essentially unique. This strengthens work of this author with Schwede on the uniqueness of models of the stable homotopy theory of spectra. As an application we show that with an added assumption about underlying model structures Margolis' axioms uniquely determine the stable homotopy category of spectra up to monoidal equivalence. Also, the equivalences constructed here give a unified construction of the known equivalences of the various symmetric monoidal categories of spectra (S-modules, $\mathscr{W}$-spaces, orthogonal spectra, simplicial functors) with symmetric spectra. The equivalences of modules, algebras and commutative algebras in these categories are also considered.

1. Introduction

The homotopy category of spectra, obtained by inverting the weak equivalences of spectra, has long been known to have a symmetric monoidal product (or tensor product) induced by the smash product [1, 26]. Recently, several categories of spectra have been constructed which have symmetric monoidal smash products even before the weak equivalences are inverted [6, 10, 11, 14]. Such categories are of interest because they facilitate the development of algebraic constructions such as ring spectra and module spectra. In each of these examples, inverting the weak equivalences recovers the standard homotopy category of spectra with the standard smash product. This raises the question of whether this is forced. In this paper we consider this question about the uniqueness properties of the monoidal product on categories of spectra and on the homotopy category of spectra.

Each of these categories of spectra is in fact a highly structured category. This structure includes a *simplicial Quillen model structure* which encodes standard homotopy theoretic constructions [16, Chapter II §2]. The symmetric monoidal product is also compatible with this model structure so that the derived product induces a symmetric monoidal product on the homotopy category where the weak equivalences have been inverted. Another common property of these model categories of spectra is that they are stable - the suspension functor is invertible up to homotopy (with inverse the loop functor); see Definition 2.4. A category with such compatible structures and a cofibrant desuspension of the unit is called a *stable simplicial monoidal model category*; see Definition 4.5. (A cofibrant desuspension of the unit exists in these categories of spectra and any model category for which the conclusion of Theorem 1.1 below holds; see Remarks 4.6 and 4.9.)

Instead of restricting to these known model categories of spectra we consider any stable simplicial monoidal model category $\mathcal{C}$. First we show that symmetric spectra, Sp^{Σ} [10], is initial among such model categories by constructing a functor from Sp^{Σ} to $\mathcal{C}$ which is simplicial, *strong monoidal* and a *left Quillen adjoint*. That is, the functor is compatible with the simplicial action, the monoidal product and the model category structure. (See [16, II §2], Definition 2.2 and Definition 2.3).

Date: February 1, 2008; 2000 AMS Math. Subj. Class.: 55U35, 55P42.
Research partially supported by an NSF grant.

Theorem 1.1. *There is a simplicial, strong monoidal, left Quillen functor from Sp^Σ to $\mathcal{C}$ for $\mathcal{C}$ any stable simplicial monoidal model category. That is, the positive stable model category on symmetric spectra is initial among the stable simplicial monoidal model categories.*

This statement is proved as Theorem 4.7. The positive stable model structure on Sp^Σ is recalled in Definition 4.1. This positive model category captures the same homotopy theory as the standard model category (i.e, they are *Quillen equivalent*, see Definition 2.3) but is initial because the sphere spectrum is not positive cofibrant; see Theorem 4.2.

Under additional assumptions on $\mathcal{C}$, the functor constructed in Theorem 1.1 is a Quillen equivalence. Hence these additional assumptions uniquely specify the models for the homotopy theory of spectra among the stable simplicial monoidal model categories. The first additional assumption here is that the unit $\mathbb{I}$ of the monoidal product in $\mathcal{C}$ is a *small, weak generator* in the homotopy category of $\mathcal{C}$. This is equivalent to asking that $[\mathbb{I}, -]^{\mathrm{Ho}(\mathcal{C})}$ commutes with coproducts and detects isomorphisms; see Definition 2.5. For example, the sphere spectrum, $\mathbb{S}$, is a small, weak generator of the homotopy category of spectra. We also ask that $[\mathbb{I}, \mathbb{I}]^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$, as holds for $\mathbb{S}$ in the homotopy category of spectra.

Theorem 1.2. *Let $\mathcal{C}$ be a stable simplicial monoidal model category. There is a monoidal Quillen equivalence from the positive stable model structure on Sp^Σ to $\mathcal{C}$ if and only if the unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$.*

This shows that up to monoidal Quillen equivalence there is a unique stable simplicial monoidal model category of spectra which satisfies the above hypotheses. Other equivalent conditions are stated in Theorem 4.8. In Section 5 this uniqueness is extended to modules, algebras and commutative algebras. Remark 4.9 shows that the monoidal Quillen equivalences constructed in Theorems 4.7, 5.2 and 5.6 recover and unify those constructed in [14] and [19] between S-modules, orthogonal spectra, $\mathscr{W}$-spaces, simplicial functors and symmetric spectra and the associated categories of modules and algebras. The conditions on the unit in Theorem 1.2 were first studied in [21]. There we considered the uniqueness of model categories of spectra but ignored the monoidal product structure.

Theorems 1.1 and 1.2 give the most highly structured uniqueness properties of the monoidal product on the model category level. Next we consider a weaker situation which is still strong enough to establish uniqueness properties of the monoidal product on the homotopy category. This less structured situation also provides an approach to Margolis' Conjecture, see Theorem 1.5 below, with fewer hypotheses than would be required using the above statements.

On the homotopy category, Corollary 3.3 shows that under weak hypotheses if there is a natural transformation $A \wedge B \longrightarrow A \wedge' B$ between two monoidal products which both have the sphere spectrum, $\mathbb{S}$, as the unit, then this transformation is an isomorphism on all objects. Thus, the main obstruction to showing that two monoidal products are equivalent is constructing a natural transformation between them.

To construct such natural transformations we consider the model categories of spectra, rather than the homotopy category. Here we consider *stable monoidal model categories*, that is, stable model categories $\mathcal{C}$ with a compatible monoidal product and a cofibrant desuspension of the unit; see Definition 6.1. We construct a functor from the homotopy category of spectra, Ho(Sp), to the homotopy category of $\mathcal{C}$, $\mathrm{Ho}(\mathcal{C})$ which induces a natural isomorphism between the smash product on Ho(Sp) and the derived product on $\mathrm{Ho}(\mathcal{C})$ (i.e, a strong monoidal functor, see Definition 2.2).

Theorem 1.3. *Let $\mathcal{C}$ be a stable monoidal model category. Then there is a strong monoidal functor from* Ho(Sp) *to* $\mathrm{Ho}(\mathcal{C})$.

This statement is proved as Theorem 6.2. As above, with added hypotheses on the unit this strong monoidal functor induces a structured monoidal equivalence between $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathrm{Sp})$.

Theorem 1.4. *Let $\mathcal{C}$ be a stable monoidal model category. There is a π_*^s-linear, triangulated, monoidal equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of spectra if and only if the unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$.*

This shows that the only monoidal product on $\mathrm{Ho}(\mathrm{Sp})$ which has an underlying model satisfying these weak hypotheses is the usual smash product. This statement and several other equivalent conditions are proved as Theorem 6.3.

We apply these results to Margolis' Conjecture from [15]. Margolis introduced axioms for a *stable homotopy category* which basically ensure that it is the structured completion of the Spanier-Whitehead category of finite CW complexes; see Definition 3.4. He then conjectured that these axioms uniquely determine the stable homotopy category of spectra. Here we add the assumption that the category has an underlying stable monoidal model category; see Definition 3.5.

Theorem 1.5. *Suppose that $\mathscr{S}$ is a stable homotopy category in the sense of* [15, Chapter 2 §1] *which has an underlying stable monoidal model category. Then $\mathscr{S}$ is monoidally equivalent to the stable homotopy category of spectra.*

Acknowledgments: The initial impetus for this paper was an observation of Hovey which appeared in an early version of [10]. Corollary 3.3 is a modification and generalization of that observation. This work continues the study begun in [21] on uniqueness properties of model categories of spectra where the monoidal product was ignored. The construction of the functors in the simplicial case, see Section 4, builds on the special cases developed in [14] and [19]. The construction of functors in the non-simplicial case, see Section 6, builds on the treatment of cosimplicial resolutions in [21]. The new ingredient is a monoidal product on cosimplicial resolutions that has not been considered before. I would also like to thank Mike Mandell and Charles Rezk for helpful suggestions during this project.

2. Model category preliminaries

In this section we recall the relevant definitions. A monoidal model category is a model category with a compatible symmetric monoidal product. Note that we do require the product to be symmetric even though that term is suppressed in the name 'monoidal model category'. The compatibility is expressed by the pushout product axiom below. This compatibility is analogous to the simplicial axiom of [16, Chapter II §2]. In particular, the product on a monoidal model category induces a derived product on the homotopy category which is symmetric monoidal. Monoidal model categories have been studied in [20] and [8]. Here, instead of requiring a closed monoidal structure, we use the weaker hypotheses that the product commutes with colimits.

Definition 2.1. A model category $\mathcal{C}$ is a *monoidal model category* if it is endowed with a symmetric monoidal structure which commutes with colimits and satisfies the following pushout product axiom and unit axiom. We denote the symmetric monoidal product by $\wedge$ and the unit by $\mathbb{I}$.

Pushout product axiom. Let $i\colon A \longrightarrow B$ and $j\colon X \longrightarrow Y$ be cofibrations in $\mathcal{C}$. Then the map

$$i \,\square\, j\colon\ A \wedge Y \cup_{A \wedge X} B \wedge X \longrightarrow B \wedge Y$$

is a cofibration which is a weak equivalence if either i or j is a weak equivalence.

Unit axiom. If the unit is not cofibrant then fix a cofibrant replacement $u\colon Q\mathbb{I} \longrightarrow \mathbb{I}$ which is a trivial fibration from a cofibrant object $Q\mathbb{I}$. Then for any cofibrant object X the map $u \wedge X\colon Q\mathbb{I} \wedge X \longrightarrow \mathbb{I} \wedge X$ is a weak equivalence.

Definition 2.2. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between symmetric monoidal categories is *lax monoidal* if there is a map $\eta\colon \mathbb{I}_{\mathcal{D}} \longrightarrow F(\mathbb{I}_{\mathcal{C}})$ and a transformation $\phi\colon FA \wedge_{\mathcal{D}} FB \longrightarrow F(A \wedge_{\mathcal{C}} B)$, natural in both variables, such that the coherence diagrams for commutativity, associativity and unital properties commute. The functor F is *strong monoidal* if η and ϕ are isomorphisms.

Next we define the appropriate equivalences of model categories and monoidal model categories.

Definition 2.3. A pair of adjoint functors between model categories is a *Quillen adjoint pair* if the right adjoint preserves trivial fibrations and fibrations between fibrant objects. This is equivalent to the usual definition [8, Definition 1.3.1] by [3, Corollary A.2]. We regard a Quillen adjoint pair as a map of model categories in the direction of the left adjoint. A Quillen adjoint pair induces adjoint total derived functors between the homotopy categories [16, Chapter I §4 Theorem 3]. A Quillen functor pair is a *Quillen equivalence* if the total derived functors are adjoint equivalences of the homotopy categories. A *monoidal Quillen equivalence* is a Quillen equivalence between monoidal model categories with a strong monoidal left adjoint functor L such that $L(Q\mathbb{I}) \longrightarrow L(\mathbb{I})$ is a weak equivalence. An equivalence of homotopy categories via strong monoidal functors is called a *monoidal equivalence*. If one functor in an adjoint equivalence is strong monoidal then so is the other, so both the left and right total derived functors of a monoidal Quillen equivalence are strong monoidal. Hence a monoidal Quillen equivalence induces a monoidal equivalence on the homotopy categories.

In this paper we actually consider only stable model categories. Recall from [16, Chapter I §2] or [8, Definition 6.1.1] that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω.

Definition 2.4. A *stable model category* is a pointed, complete and cocomplete category with a model category structure for which the functors Ω and Σ on the homotopy category are inverse equivalences.

Certain extra structures on the homotopy category of a stable model category are key here. The homotopy category is naturally a triangulated category [25]. The suspension functor defines the shift functor and the cofiber sequences of [16, Chapter I §3] define the distinguished triangles (the fiber sequences agree up to sign [8, Theorem 7.1.11]); see [8, Proposition 7.1.6] for more details. We have required a stable model category to have all limits and colimits so that its homotopy category has infinite sums and products. The homotopy category of a stable model category also has a natural action of the ring π_*^s of stable homotopy groups of spheres [21, Construction 2.3]. If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is the left adjoint of a Quillen adjoint pair between stable model categories, then the total left derived functor $LF : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ of F is π_*^s-linear and an exact functor [21, Lemma 6.1], [8, Proposition 6.4.1].

For objects A and X of a triangulated category $\mathcal{T}$ we denote by $[A, X]_*^{\mathcal{T}}$ the graded abelian homomorphism group defined by $[A, X]_m^{\mathcal{T}} = [A[m], X]^{\mathcal{T}}$ for $m \in \mathbb{Z}$, where $A[m]$ is the m-fold shift of A. If $\mathcal{T}$ is a π_*^s-triangulated category, then the groups $[A, X]_*^{\mathcal{T}}$ form a graded π_*^s-module.

Definition 2.5. An object G of a triangulated category $\mathcal{T}$ is called a *weak generator* if it detects isomorphisms; i.e., a map $f : X \longrightarrow Y$ is an isomorphism if and only if it induces an isomorphism between the graded abelian homomorphism groups $\mathcal{T}(G, X)_*$ and $\mathcal{T}(G, Y)_*$. An object G of $\mathcal{T}$

is *small* if for any family of objects $\{A_i\}_{i\in I}$ whose coproduct exists the canonical map

$$\bigoplus_{i\in I} \mathcal{T}(G, A_i) \longrightarrow \mathcal{T}(G, \coprod_{i\in I} A_i)$$

is an isomorphism.

3. Margolis' uniqueness conjecture

In this section we apply our monoidal uniqueness results to Margolis' conjecture about the uniqueness of the stable homotopy category. Margolis introduced axioms for a stable homotopy category in [15]. He then conjectured that these axioms uniquely specify the stable homotopy category of spectra up to a monoidal, triangulated equivalence of categories. In [21], any stable homotopy category satisfying Margolis' axioms and having an underlying model category was shown to be triangulated equivalent to the stable homotopy category of spectra. Here we strengthen that result to a monoidal, triangulated equivalence.

First we consider a more general setting than Margolis' stable homotopy categories. The following proposition shows that under weak hypotheses a lax monoidal functor between two monogenic, monoidal, triangulated categories is strong monoidal.

Definition 3.1. A *monogenic, monoidal, triangulated category* is a triangulated category $\mathcal{T}$ with arbitrary coproducts and with a symmetric monoidal, bi-exact smash product $\wedge$ which commutes with suspensions and coproducts such that the unit $\mathbb{I}$ is a small, weak generator.

Proposition 3.2. *Assume $(\mathcal{T}, \wedge, \mathbb{I})$ and $(\mathcal{T}', \wedge', \mathbb{I}')$ are two monogenic, monoidal, triangulated categories. Suppose that $F\colon \mathcal{T} \longrightarrow \mathcal{T}'$ is a lax monoidal, exact functor with unit map $\eta\colon \mathbb{I}' \longrightarrow F(\mathbb{I})$ and natural transformation $\phi\colon FA \wedge' FB \longrightarrow F(A\wedge B)$. If F commutes with coproducts, η is an isomorphism and $\phi\colon F\mathbb{I}\wedge' F\mathbb{I} \longrightarrow F(\mathbb{I}\wedge\mathbb{I})$ is an isomorphism, then F is strong monoidal.*

Proof. Consider the subcategory of objects A in $\mathcal{T}$ such that $\phi\colon FA \wedge' F\mathbb{I} \longrightarrow F(A\wedge\mathbb{I})$ is an isomorphism. By the assumptions on F, $\wedge$ and $\wedge'$, both source and target commute with triangles and coproducts. So this subcategory is a localizing subcategory which contains $\mathbb{I}$. Since $\mathbb{I}$ is a small, weak generator it follows that this subcategory is the whole category. This follows from [9, Theorem 2.3.2]; see also [22, Lemma 2.2.1]. Now fix any A and consider the subcategory of objects B in $\mathcal{T}$ such that $\phi\colon FA\wedge' FB \longrightarrow F(A\wedge B)$ is an isomorphism. Again this is a localizing subcategory which contains $\mathbb{I}$, and hence is the whole category. Thus, ϕ is an isomorphism for any A and B. $\square$

Since the stable homotopy category of spectra is a monogenic, monoidal, triangulated category, this gives the following corollary.

Corollary 3.3. *Assume that $\wedge$ and $\wedge'$ are two monogenic, monoidal, triangulated structures on the homotopy category of spectra, both with unit the sphere spectrum, $\mathbb{S}$. If the identity functor is lax monoidal and the unit map η and the natural transformation ϕ evaluated on the unit are isomorphisms, then the identity functor gives a monoidal equivalence between these two structures.*

So the only obstruction to showing that the smash product of spectra is unique up to monoidal equivalence on the homotopy category is constructing a natural transformation between any two monoidal products. Our solution is to assume there is an underlying stable monoidal model category We state this result for Margolis' stable homotopy categories.

Definition 3.4. A *stable homotopy category* in the sense of [15, Chapter 2 §1] is a monogenic, monoidal, triangulated category $\mathscr{S}$ with an exact and strong symmetric monoidal equivalence

$R : \mathcal{SW}_{\mathrm{f}} \longrightarrow \mathscr{S}^{\mathrm{small}}$ between the Spanier-Whitehead category of finite CW-complexes ([24], [15, Chapter 1, §2]) and the full subcategory of small objects in $\mathscr{S}$.

As shown in [21, Section 3], such an equivalence induces a π_*^s-linear structure on the triangulated category $\mathscr{S}$. In fact, we could weaken the definition above to only require that there is such an equivalence R with the full subcategory of $\mathcal{SW}$ on the spheres S^n for n an integer.

Definition 3.5. A stable homotopy category $\mathscr{S}$ has an *underlying stable monoidal model category* if there is a monoidal, π_*^s-linear, exact equivalence $\Phi\colon \mathscr{S} \longrightarrow \mathrm{Ho}(\mathcal{C})$ with $\mathcal{C}$ a stable monoidal model category; see Definition 6.1.

Proof of Theorem 1.5. Since $\mathscr{S}$ has an underlying stable monoidal model category, there is an equivalence $\Phi\colon \mathscr{S} \longrightarrow \mathrm{Ho}(\mathcal{C})$ with all of the properties mentioned in Definition 3.5. Since the properties of a small, weak generator are determined on the homotopy category level, the image $X \in \mathrm{Ho}(\mathcal{C})$ under Φ of the unit object in $\mathscr{S}$ is a small weak generator of the homotopy category of $\mathcal{C}$. Because the equivalence Φ is monoidal and π_*^s-linear, X is isomorphic to the unit and satisfies the hypotheses on the unit in Theorem 1.4. Thus, the homotopy category of $\mathcal{C}$, and hence also $\mathscr{S}$, is monoidally equivalent to the ordinary stable homotopy category of spectra. $\square$

4. Simplicial monoidal model categories

Here we construct a Quillen adjoint pair from the positive stable model category on Sp^Σ to $\mathcal{C}$. Then, under additional hypotheses on the unit, this produces a monoidal Quillen equivalence from Sp^Σ to $\mathcal{C}$. First we recall the positive model structure from [14, Section 14].

Definition 4.1. In the positive stable model structure on Sp^Σ a map f is a weak equivalence if it is a stable equivalence, [10, 14]. The map f is a positive trivial fibration if f_n is a trivial fibration for $n > 0$. The positive cofibrations and positive fibrations are then determined by the respective right and left lifting properties with respect to the trivial fibrations and the trivial cofibrations.

In [14] only symmetric spectra over topological spaces are considered, but the arguments can be easily modified to apply to symmetric spectra over simplicial sets. The identity functor from the usual to the positive structure is a right Quillen functor since (trivial) fibrations are in particular positive (trivial) fibrations.

Theorem 4.2. [14, Theorem 14.2, Proposition 14.6] *The positive stable model structure on Sp^Σ forms a stable, monoidal model category. The identity functor induces a monoidal Quillen equivalence from the positive stable model structure to the usual stable model structure on Sp^Σ.*

Denote the unit in Sp^Σ by $\mathbb{S}$. Note $\mathbb{S}$ is not cofibrant in the positive stable model category. To fix its cofibrant replacement for the unit axiom of the monoidal model category structure, first consider the nth evaluation functor Ev_n on symmetric spectra which lands in Σ_n-equivariant spaces. For X a Σ_n-space, the left adjoint F'_n is defined by $(F'_nX)_k \cong \Sigma_k \wedge_{\Sigma_n \times \Sigma_{k-n}} (X \wedge S^{k-n})$. This is a slight variant of the free functor F_n studied in [10]. Note that $F'_1 \cong F_1$ and $F'_0 \cong F_0$. Then define the cofibrant replacement of $\mathbb{S}$ as the weak equivalence $Q\mathbb{S} = F'_1S^1 \longrightarrow F_0S^0 = \mathbb{S}$ induced by the identity map in level one.

Proposition 4.3. *The fibrant objects in the positive stable model structure are the positive Ω-spectra. That is, X is fibrant if X_n is fibrant for $n > 0$ and $X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence for $n > 0$. A map f between positive Ω-spectra is a fibration if each f_n is a fibration for $n > 0$.*

Proof. The description of the fibrant objects follows from [14, Theorem 14.2]. The description of the fibrations follows from the fact that the positive stable model structure is a localization of the positive level model structure [14, Theorem 14.1]. In a localized model structure the fibrations

between fibrant objects are the fibrations in the original model structure. So here they are the positive level fibrations. This statement also follows from the positive variants of [10, Lemma 3.4.12] or [14, Proposition 9.5]. □

We now define a stable simplicial monoidal model category. As mentioned in the introduction, here we require the following technical hypothesis on the unit which may not be required in other definitions of stable simplicial monoidal model categories but is needed here; see Remark 4.9.

Definition 4.4. A *cofibrant desuspension of the unit* is a cofibrant object $\mathbb{I}_c^{-1}$ with a weak equivalence $\eta\colon\ \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

Recall that a monoidal model category is a model category $\mathcal{C}$ with a symmetric monoidal product that is compatible with the model structure; see Definition 2.1. Similarly, a *simplicial model category* is a model category with a compatible action of simplicial sets. A simplicial functor is a functor that is compatible with this structure. See [16, Chapter II §1, 2].

Definition 4.5. A *stable simplicial monoidal model category* is a category $\mathcal{C}$ with a stable, simplicial model structure, a monoidal model structure and a cofibrant desuspension of the unit such that the simplicial action commutes with the monoidal product. That is, for X, Y in $\mathcal{C}$ and K in $\mathcal{S}_*$ there are natural coherent isomorphisms $(X \wedge Y) \otimes K \cong X \wedge (Y \otimes K)$.

Remark 4.6. If the unit $\mathbb{I}$ in $\mathcal{C}$ is fibrant, then a cofibrant desuspension exists. Since $\mathcal{C}$ is stable, there is a cofibrant object X whose suspension in the homotopy category is isomorphic to $\mathbb{I}$. Since $\mathcal{C}$ is simplicial and X is cofibrant its suspension is modeled by $X \otimes S^1$. Since $X \otimes S^1$ is cofibrant and $\mathbb{I}$ is fibrant the isomorphism in the homotopy category is realized by some weak equivalence in $\mathcal{C}$.

A cofibrant desuspension of the unit exists in every known symmetric monoidal model category of spectra. In the diagram categories of spectra investigated in [14] and their simplicial analogues [10, 11], (orthogonal spectra, symmetric spectra, and simplicial functors or $\mathscr{W}$-spaces) the cofibrant desuspension can be chosen as the object denoted $F_1 S^0$, with the weak equivalence $\eta\colon\ F_1 S^1 \longrightarrow F_0 S^0$; see [14, Definition 1.3, Remark 4.7]. The S-modules of [6] are all fibrant, so the previous paragraph applies.

Theorem 4.7. *Let $\mathcal{C}$ be a stable simplicial monoidal model category. Then there exists a Quillen adjoint functor pair from the positive stable model structure on Sp^Σ to $\mathcal{C}$, $\mathbb{I} \wedge -\colon\ Sp^\Sigma \longrightarrow \mathcal{C}$ and $\mathrm{Hom}(\mathbb{I}, -)\colon\ \mathcal{C} \longrightarrow Sp^\Sigma$. These functors are simplicial, the left adjoint $\mathbb{I} \wedge -$ is strong monoidal, and $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence.*

Remark 4.9 below shows that the existence of such a Quillen adjoint pair implies the existence of a cofibrant desuspension. Adding conditions on the unit in $\mathcal{C}$ shows this Quillen adjoint pair is a Quillen equivalence.

Theorem 4.8. *Let $\mathcal{C}$ be a stable simplicial monoidal model category. The following conditions are equivalent:*

1. *There is a π_*^s-linear triangulated equivalence from the homotopy category of Sp^Σ to the homotopy category of $\mathcal{C}$ which takes the unit $\mathbb{I}$ of the monoidal product in $\mathcal{C}$ to the unit $\mathbb{S}$ of Sp^Σ.*
2. *The unit, $\mathbb{I}$, is a small weak generator for which $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by the identity map of $\mathbb{I}$.*
3. *There is a simplicial, monoidal Quillen equivalence from the positive stable model structure on Sp^Σ to $\mathcal{C}$.*

4. *There is a zig-zag of monoidal Quillen equivalences between the usual stable model structure on Sp^{Σ} and $\mathcal{C}$.*

Proof. Condition (1) implies condition (2) since the properties of $\mathbb{I}$ mentioned in (2) hold for $\mathbb{S}$ and are determined by the π_*^s-linear triangulated homotopy category. Condition (3) implies condition (4) because the positive stable model structure is monoidally Quillen equivalent to the usual stable model structure on Sp^{Σ} by Theorem 4.2. Since Quillen functors induce π_*^s-linear triangulated functors on the homotopy categories by [8, Proposition 6.4.1] and [21, Lemma 6.1] and monoidal functors preserve the unit, condition (4) implies condition (1).

Next we show that given condition (2) the simplicial Quillen adjoint pair constructed in Theorem 4.7 is a Quillen equivalence. Since $\mathbb{I} \wedge -$ is strong monoidal, $\mathbb{I} \wedge \mathbb{S} \cong \mathbb{I}$. Also, $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence, so $\mathbb{I} \wedge^L \mathbb{S} \cong \mathbb{I}$. The total derived functor $\mathbb{I} \wedge^L -$ is exact by [8, Proposition 6.4.1]. So $\mathbb{I} \wedge^L \mathbb{S}[n] \cong \mathbb{I}[n]$ where $X[n]$ denotes the nth shift of X for any integer n. This isomorphism and the derived adjunction for $\mathbb{I} \wedge^L -$ and $\mathrm{RHom}(\mathbb{I}, -)$ produce the following natural isomorphisms

$$\pi_* \mathrm{RHom}(\mathbb{I}, Y) \cong [\mathbb{S}[*], \mathrm{RHom}(\mathbb{I}, Y)] \cong [\mathbb{I}, Y]_*^{\mathrm{Ho}(\mathcal{C})}.$$

Since $\mathbb{I}$ is a weak generator, $\mathrm{RHom}(\mathbb{I}, -)$ detects isomorphisms. So to show that this pair is a Quillen equivalence we need to show that for any symmetric spectrum A the unit of the adjunction $A \longrightarrow \mathrm{RHom}(\mathbb{I}, \mathbb{I} \wedge^L A)$ is an isomorphism. Consider the full subcategory $\mathcal{T}$ of such objects. For $A = \mathbb{S}$ in homotopy this map is the map $[\mathbb{S}, \mathbb{S}]_* \longrightarrow [\mathbb{I}, \mathbb{I}]_*$ induced by $\mathbb{I} \wedge^L -$. This map of free π_*^s-modules takes the identity map of $\mathbb{S}$ to the identity map of $\mathbb{I}$. Hence it is also an isomorphism by condition (2). So $\mathbb{S}$ is contained in $\mathcal{T}$. Since $\mathbb{I}$ is small, $\mathrm{RHom}(\mathbb{I}, -)$ commutes with coproducts by the display above. Hence, since left adjoints commute with coproducts and total derived functors between stable model categories are exact, the composite $\mathrm{RHom}(\mathbb{I}, \mathbb{I} \wedge^L -)$ is an exact functor which commutes with coproducts. So $\mathcal{T}$ is a localizing subcategory which contains the generator $\mathbb{S}$ of symmetric spectra. Hence $\mathcal{T}$ is the whole category. Thus, these derived functors induce an equivalence of homotopy categories. □

Proof of Theorem 4.7. We first construct the functor $\mathrm{Hom}(\mathbb{I}, -)\colon \mathcal{C} \longrightarrow Sp^{\Sigma}$. Let $\mathbb{I}_c^{-1}$ be a cofibrant desuspension of the unit in $\mathcal{C}$ with a weak equivalence $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$. Let $\mathbb{I}_c^{-n} = (\mathbb{I}_c^{-1})^{\wedge n}$ be the n-fold smash product of $\mathbb{I}$ where $X^0 = \mathbb{I}$. Notice that in general $\mathbb{I}_c^0$ is not cofibrant. For Y in $\mathcal{C}$, define the nth level of $\mathrm{Hom}(\mathbb{I}, Y)_n$ to be the simplicial mapping space $\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, Y)$. The symmetric group on n letters acts on $\mathbb{I}_c^{-n}$ by permuting the factors and hence also acts on $\mathrm{Hom}(\mathbb{I}, Y)_n$. The structure map

$$\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, Y) \longrightarrow \Omega^m \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)}, Y) \cong \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)} \otimes S^m, Y)$$

is induced by $\mathrm{map}_{\mathcal{C}}(\sigma, Y)$ with σ defined as

$$\sigma_{n,m}\colon \mathbb{I}_c^{-(n+m)} \otimes S^m \cong \mathbb{I}_c^{-n} \wedge (\mathbb{I}_c^{-1} \otimes S^1)^m \xrightarrow{\mathrm{id} \wedge (\eta)^m} \mathbb{I}_c^{-n} \wedge (\mathbb{I})^m \cong \mathbb{I}_c^{-n}.$$

Since the adjoint of $\mathrm{map}_{\mathcal{C}}(\sigma, Y)$ is $\Sigma_n \times \Sigma_m$ equivariant, this makes $\mathrm{Hom}(\mathbb{I}, Y)$ into a symmetric spectrum. Here we have used the fact that the simplicial action and the monoidal product commute. $\mathrm{Hom}(\mathbb{I}, -)$ is an example of a categorical construction described in [13, I.2].

Since $\mathcal{C}$ is a simplicial model category and $\mathbb{I}_c^{-n}$ is cofibrant for $n > 0$, $\mathrm{Hom}(\mathbb{I}, -)$ of a (trivial) fibration is a (trivial) fibration in levels $n > 0$. Since $\mathbb{I}_c^{-1} \otimes S^1$ is cofibrant η factors as $\mathbb{I}_c^{-1} \otimes S^1 \longrightarrow Q\mathbb{I} \longrightarrow \mathbb{I}$ where $Q\mathbb{I} \longrightarrow \mathbb{I}$ is the fixed cofibrant replacement of $\mathbb{I}$ given in the monoidal model structure on $\mathcal{C}$. Since $\mathcal{C}$ is monoidal and η is a weak equivalence, $\sigma_{n,1}$ is a weak equivalence between cofibrant objects for $n > 0$. Hence $\mathrm{Hom}(\mathbb{I}, -)$ takes a fibrant object to a positive Ω-spectrum, which is a fibrant object in the positive stable model structure. Thus $\mathrm{Hom}(\mathbb{I}, -)$

takes trivial fibrations to positive trivial fibrations and fibrations to positive fibrations between positive fibrant objects by Proposition 4.3. So $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint.

Next we consider the left adjoint $\mathbb{I} \wedge -\colon Sp^\Sigma \longrightarrow \mathcal{C}$. Using the definition of $F'_n X$, $\mathbb{I} \wedge F'_n X$ is isomorphic to $\mathbb{I}_c^{-n} \otimes_{\Sigma_n} X$ since both corepresent the functor which takes Y in $\mathcal{C}$ to the space of Σ_n-equivariant maps from X to $\mathrm{Hom}(\mathbb{I}, Y)_n$. So $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is isomorphic to the weak equivalence $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

To evaluate $\mathbb{I} \wedge -$ on an arbitrary symmetric spectrum A, note that A can be built as the coequalizer of the following diagram:

$$\bigvee_n F'_{n+1}(\Sigma_{n+1} \wedge_{\Sigma_n} (A_n \wedge S^1)) \implies \bigvee_n F'_n A_n$$

Here one map is induced by the map $A_n \wedge S^1 \longrightarrow A_{n+1}$ and the other is induced by smashing $F'_n A_n$ with the map $F'_1 S^1 \longrightarrow F'_0 S^0$ which is the adjoint of the identity map on S^1 in level one. Since $\mathbb{I} \wedge -$ must commute with colimits, $\mathbb{I} \wedge A$ is defined as the coequalizer of the diagram:

$$\bigvee_n \mathbb{I}_c^{-(n+1)} \otimes_{\Sigma_n} (A_n \wedge S^1) \implies \bigvee_n \mathbb{I}_c^{-n} \otimes_{\Sigma_n} A_n$$

Again the first map is induced by $A_n \wedge S^1 \longrightarrow A_{n+1}$ and the second map uses the fact that the simplicial action and monoidal product in $\mathcal{C}$ commute to give the isomorphism

$$\mathbb{I}_c^{-(n+1)} \otimes_{\Sigma_n} (A_n \wedge S^1) \cong (\mathbb{I}_c^{-n} \otimes_{\Sigma_n} A_n) \wedge (\mathbb{I}_c^{-1} \otimes S^1)$$

along with the map $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$.

Next we consider the monoidal properties of these adjoint functors. First $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal; since the simplicial action and monoidal product commute the product of maps induces $\mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-n}, A) \wedge \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-m}, B) \longrightarrow \mathrm{map}_{\mathcal{C}}(\mathbb{I}_c^{-(n+m)}, A \wedge B)$. These fit together to give a natural map $\mathrm{Hom}(\mathbb{I}, A) \wedge \mathrm{Hom}(\mathbb{I}, B) \longrightarrow \mathrm{Hom}(\mathbb{I}, A \wedge B)$. The unit map $F'_0 S^0 = \mathbb{S} \longrightarrow \mathrm{Hom}(\mathbb{I}, \mathbb{I})$ is given by sending the non-base point of S^0 to the identity map of $\mathbb{I}$ in simplicial degree zero of $\mathrm{Hom}(\mathbb{I}, \mathbb{I})_0 = \mathrm{map}_{\mathcal{C}}(\mathbb{I}, \mathbb{I})$.

The left adjoint of a lax monoidal functor is automatically lax comonoidal. That is, there are structure maps in the opposite direction of a lax monoidal functor; see the display below. The adjoint of the unit map is an isomorphism $\mathbb{I} \wedge \mathbb{S} \longrightarrow \mathbb{I}$. Denote the adjoint pair by L and R. Then the counit and unit of the adjunction and the lax monoidal structure of R give

$$L(A \wedge B) \longrightarrow L(RLA \wedge RLB) \longrightarrow LR(LA \wedge LB) \longrightarrow LA \wedge LB.$$

Here in fact $L = \mathbb{I} \wedge -$ is strong monoidal because this map is an isomorphism. To show this we only need to consider the special case where $A = F'_n X$ and $B = F'_m Y$ for X a Σ_n-space and Y a Σ_m-space since the general case follows by using the coequalizer diagrams above. Then

$$L(A \wedge B) = LF'_{n+m}(\Sigma_{n+m} \wedge_{\Sigma_n \times \Sigma_m} X \wedge Y) \cong \mathbb{I}_c^{-(n+m)} \otimes_{\Sigma_n \times \Sigma_m} X \wedge Y.$$

Again commuting the simplicial action and the monoidal product shows this last term is isomorphic via the transformation displayed above to $(\mathbb{I}_c^{-n} \otimes_{\Sigma_n} X) \wedge (\mathbb{I}_c^{-m} \otimes_{\Sigma_m} Y) = LA \wedge LB$. These monoidal properites also follow from the more general treatment in [13, I.2].

Finally, these adjoint functors $\mathrm{Hom}(\mathbb{I}, -)$ and $\mathbb{I} \wedge -$ are simplicial functors. This follows by various adjunctions from the isomorphism $\mathrm{Hom}(\mathbb{I}, Y^K) \cong \mathrm{Hom}(\mathbb{I}, Y)^K$ given by the simplicial structure on $\mathcal{C}$. □

Remark 4.9. If there is a Quillen adjoint pair from the positive stable model structure on Sp^Σ to $\mathcal{C}$ with a strong monoidal, simplicial left adjoint L which takes $Q\mathbb{S} \longrightarrow \mathbb{S}$ to a weak equivalence, then a cofibrant desuspension of the unit exists. Set $\mathbb{I}_c^{-1} = L(F'_1 S^0)$. The map $\eta\colon \mathbb{I}_c^{-1} \otimes S^1 \longrightarrow \mathbb{I}$ is then given by $L(F'_1 S^0) \otimes S^1 \xrightarrow{\varphi} L(F'_1 S^1 = Q\mathbb{S}) \longrightarrow L(\mathbb{S})$ where φ is induced by the simplicial

structure on L. The second map is a weak equivalence by assumption. The first map is a weak equivalence because it is the cofiber of the weak equivalence $L(F_1S^0)\otimes\Delta[1]_+ \xrightarrow{\varphi} L(F_1S^0\otimes\Delta[1]_+)$ by the isomorphism $L(F_1S^0)\otimes(S^0\vee S^0) \xrightarrow{\varphi} L(F_1S^0\otimes(S^0\vee S^0))$.

This also gives a procedure for recovering the known equivalences between symmetric monoidal model categories of spectra as $\mathbb{I}\wedge -$ and $\mathrm{Hom}(\mathbb{I},-)$ for some choice of a cofibrant desuspension of the unit. For the monoidal functors constructed in [14] ($\mathbb{P}$ and $\mathbb{U}$ between orthogonal spectra and symmetric spectra and between $\mathscr{W}$-spaces and symmetric spectra), the chosen desuspension of the unit is $\mathbb{P}(F_1S^0)\cong F_1S^0$ [14, Definition 1.3, Remark 4.7]. The monoidal functors (Λ and Φ) between S-modules and symmetric spectra as defined in [19] are isomorphic to $\mathbb{I}\wedge -$ and $\mathrm{Hom}(\mathbb{I},-)$ with $\Lambda(F_1S^0)\cong S_c^{-1}$ the chosen desuspension of the unit.

Remark 4.10. If $\mathcal{C}$ is a cofibrantly generated, proper, stable model category then [18, Proposition 4.4] shows that $\mathcal{C}$ is Quillen equivalent to a simplicial model category structure on the category of simplicial objects, $\mathcal{C}^{\Delta^{\mathrm{op}}}$. If the product on $\mathcal{C}$ commutes with coproducts then the level prolongation of the product commutes with the simplicial action. Using [7, Proposition 16.11.1, Theorem 16.4.2], one can show that if $\mathcal{C}$ is a monoidal model category then the simplicial model category from [18] is also monoidal. Hence, under these conditions, one can apply the constructions in this section to the stable simplicial monoidal model category on $\mathcal{C}^{\Delta^{\mathrm{op}}}$. This remark can also be applied if $\mathcal{C}$ is simplicial and the product does not commute with the simplicial action but does commute with coproducts. We treat the non-simplicial case in even more generality in Section 6.

5. Modules and Algebras

In this section, we show that the functors constructed in Theorem 4.7 induce Quillen adjoint pairs on modules, algebras and commutative algebras. Since $\mathbb{I}\wedge -$ is strong symmetric monoidal and $\mathrm{Hom}(\mathbb{I},-)$ is lax symmetric monoidal, these functors restrict to adjoint functors on subcategories of modules and algebras. Since we want the restriction of $\mathrm{Hom}(\mathbb{I},-)$ to be a right Quillen adjoint, we assume that in the model structures on categories of modules or algebras over $\mathcal{C}$ a morphism is a weak equivalence or fibration if it is one in the underlying model structure on $\mathcal{C}$. The next proposition states sufficient conditions for this assumption to hold for modules and associative algebras. We treat commutative algebras separately in the second part of this section.

Proposition 5.1. [20, Theorem 4.1] *Assume $\mathcal{C}$ is a cofibrantly generated, monoidal model category that satisfies the monoid axiom* [20, Definition 3.3]. *If the objects in $\mathcal{C}$ satisfy certain smallness conditions* [20, Lemma 2.3]*, then the category of left R-modules (for a fixed monoid R) and the category of R-algebras (for a fixed commutative monoid R) are model categories with fibrations and weak equivalences determined in $\mathcal{C}$.*

Theorem 5.2. *Let $\mathcal{C}$ be a stable simplicial monoidal model category with a monoidal Quillen equivalence from Sp^{Σ} to $\mathcal{C}$ (or any equivalent condition from Theorem 4.8) such that the conclusions of the previous proposition hold. If $\mathbb{I}\wedge -$ preserves weak equivalences between stably cofibrant symmetric spectra, then $\mathbb{I}\wedge -$ and $\mathrm{Hom}(\mathbb{I},-)$ induce a Quillen equivalence*

1. *from the positive stable model category of R-modules for R a cofibrant symmetric ring spectrum to $(\mathbb{I}\wedge R)$-modules, and*
2. *from the positive stable model category of R-algebras for R a commutative symmetric ring spectrum which is cofibrant as a symmetric spectrum to $(\mathbb{I}\wedge R)$-algebras.*

These statements also hold with the usual stable model category replacing the positive one if $\mathbb{I}$ is cofibrant.

Since $\mathbb{S}$ is cofibrant as a symmetric spectrum the second statement implies that the category of symmetric ring spectra, $\mathbb{S}$-algebras, and the category of monoids in $\mathcal{C}$, $\mathbb{I}$-algebras, are Quillen equivalent.

Remark 5.3. The hypothesis that $\mathbb{I} \wedge -$ preserves weak equivalences between stably cofibrant symmetric spectra is satisfied when $\mathcal{C}$ is any one of the symmetric monoidal model categories of orthogonal spectra, $\mathscr{W}$-spaces [14], simplicial functors [11], or S-modules [6]. Since the unit is cofibrant in the first three cases, this follows from the next proposition. This holds in the case of S-modules by [19, Theorem 3.1] and the fact that $\mathrm{Hom}(\mathbb{I}, -)$ detects and preserves weak equivalences.

Proposition 5.4. *If $\mathbb{I}$ is cofibrant and $\mathcal{C}$ is a stable simplicial monoidal model category which satisfies any of the equivalent conditions of Theorem 4.8, then $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ form a Quillen equivalence from the usual stable model category of symmetric spectra to $\mathcal{C}$. Hence $\mathbb{I} \wedge -$ preserves weak equivalences between cofibrant symmetric spectra.*

Proof. If $\mathbb{I}$ is cofibrant, then $\mathrm{Hom}(\mathbb{I}, -)_0 = \mathrm{map}(\mathbb{I}, -)$ also preserves (trivial) fibrations. Hence $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint functor from $\mathcal{C}$ to the usual stable model category of symmetric spectra. The statements follow from the same proof as given in Theorem 4.8. □

Proof of Theorem 5.2. Since the (trivial) fibrations in the categories of $(\mathbb{I} \wedge R)$-modules and $(\mathbb{I} \wedge R)$-algebras are determined on the underlying category, the restriction of $\mathrm{Hom}(\mathbb{I}, -)$ in both cases is still a right Quillen adjoint functor to the positive model structure. Since $\mathbb{I}$ is assumed to be a weak generator by condition (2) of Theorem 4.8, $\mathrm{Hom}(\mathbb{I}, -)$ preserves and detects weak equivalences. So by [10, Lemma 4.1.7] we only need to show that $\psi_A\colon A \longrightarrow \mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge A)^f)$ is a weak equivalence for A a positive cofibrant object in R-modules or R-algebras where $(\mathbb{I} \wedge A)^f$ is a fibrant replacement. Since fibrations are determined on the underlying category, a fibrant replacement as a module or algebra restricts to a fibrant replacement in $\mathcal{C}$.

Under the given conditions on R, if A is cofibrant in the positive model category of R-modules or R-algebras then A is cofibrant as a symmetric spectrum. By [14, Proposition 14.6] the identity functor from the positive stable model structure on R-modules to the usual stable model structure on R-modules is a Quillen left adjoint. So if A is a positive cofibrant R-module then it is a cofibrant R-module. Since R is assumed to be cofibrant as a symmetric ring spectrum it is cofibrant as a symmetric spectrum by [14, Theorem 12.1(v)]. Hence, by [14, Theorem 12.1(ii)], A is cofibrant as a symmetric spectrum. Again by [14, Proposition 14.6], if A is a positive cofibrant R-algebra, then it is a cofibrant R-algebra. Then by [14, Theorem 12.1 (ii), (v)] it follows that A is cofibrant as a symmetric spectrum.

We now show that ψ_B is a weak equivalence for B any cofibrant symmetric spectrum. It then follows that $\mathbb{I} \wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ restrict to Quillen equivalences on the positive stable model categories of R-modules and R-algebras. The proof of Theorem 4.8 shows that ψ_A is a weak equivalence for A any positive cofibrant symmetric spectrum. Given a cofibrant symmetric spectrum B, choose a positive cofibrant replacement $\phi\colon cB \longrightarrow B$. Since $\mathbb{I} \wedge -$ preserves weak equivalences between cofibrant objects and positive cofibrant objects are cofibrant, $\mathbb{I} \wedge cB \longrightarrow \mathbb{I} \wedge B$ is a weak equivalence. Then one can choose fibrant replacements and a lift $(\mathbb{I} \wedge \phi)^f$ so that $\psi_B \circ \phi = \mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge \phi)^f) \circ \psi_{cB}$. Thus, ψ_B is a weak equivalence, since $\mathrm{Hom}(\mathbb{I}, -)$ preserves weak equivalences between fibrant objects and ϕ and ψ_{cB} are weak equivalences.

If $\mathbb{I}$ is cofibrant then $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint functor to the usual stable model structures. So the last statement follows similarly. □

For the commutative algebra case we need several more assumptions. These assumptions are satisfied in each of the known examples of equivalences of commutative algebra spectra [14, §16]

and [19, Theorem 5.1]. Let $PX = \vee_{i\geq 0} X^{(i)}/\Sigma_i$ be the monad on $\mathcal{C}$ which defines the commutative ring objects (or more properly, *monoids*) in $\mathcal{C}$. Here $X^{(i)}$ denotes the ith smash power with Σ_i permuting the factors and $X^{(0)} = \mathbb{I}$. To ensure that $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint here we require the weak equivalences and fibrations in the model category of commutative monoids in $\mathcal{C}$ to be maps which are underlying weak equivalences or fibrations in $\mathcal{C}$. Unlike associative algebras and modules, criteria for the existence of such a model category on commutative monoids in $\mathcal{C}$ do not exist in the current literature. Another one of our assumptions here is that the quotient map from the extended power to the symmetric power, $\Phi_i\colon\ E\Sigma_i \wedge_{Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$, is a weak equivalence for cofibrant objects X in $\mathcal{C}$. Since the monad P does not necessarily preserve weak equivalences, this is likely to be one of the criteria required for constructing a model category on commutative monoids.

Hypotheses 5.5. *Let $\mathcal{C}$ be a stable simplicial monoidal model category such that*

1. *for any commutative ring R' in $\mathcal{C}$ the commutative R'-algebras in $\mathcal{C}$ form a model category with a fibrant replacement functor and with fibrations and weak equivalences the underlying fibrations and weak equivalences in $\mathcal{C}$,*
2. $\Phi_i\colon\ E\Sigma_i \wedge_{Sigma_i} X^{(i)} \longrightarrow X^{(i)}/\Sigma_i$ *is a weak equivalence for cofibrant objects X in $\mathcal{C}$ and*
3. *there is a monoidal Quillen equivalence from Sp^{Σ} to $\mathcal{C}$ (or any other equivalent condition from Theorem 4.8).*

These hypotheses hold for the positive stable model categories on orthogonal spectra and symmetric spectra by [14, 10.4, 15.1, 15.2 and 15.5] and they hold for S-modules by [6, III.5.1, VI.4.8] and [19]. These hypotheses are more subtle than those for modules and algebras; for example, the first hypothesis does not hold for the usual stable model category on symmetric spectra; see [14, Section 14].

Theorem 5.6. *Assume $\mathcal{C}$ satisfies Hypotheses 5.5. Then $\mathbb{I}\wedge -$ and $\mathrm{Hom}(\mathbb{I}, -)$ induce a Quillen equivalence*

1. *from the commutative symmetric ring spectra to the commutative rings in $\mathcal{C}$ and*
2. *from the commutative R-algebras for R a cofibrant commutative symmetric ring spectrum to commutative $(\mathbb{I}\wedge R)$-algebras.*

Proof. The first statement is a special case of the second with $R = \mathbb{S}$ and $\mathbb{I}\wedge\mathbb{S} \cong \mathbb{I}$. Since the weak equivalences and fibrations are determined on the underlying category the restriction of $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen adjoint functor. Since the equivalent conditions in Theorem 4.8 hold, $\mathrm{Hom}(\mathbb{I}, -)$ preserves and detects weak equivalences between fibrant objects. By [10, Lemma 4.1.7] it thus suffices to show that $\psi_A\colon\ A \longrightarrow \mathrm{Hom}(\mathbb{I}, (\mathbb{I}\wedge A)^f)$ is a weak equivalence for cofibrant commutative R-algebras where $(-)^f$ is the given fibrant replacement functor. Since fibrations are determined on the underlying category a fibrant replacement as a commutative R-algebra restricts to a fibrant replacement in $\mathcal{C}$. We first consider the case where $R = \mathbb{S}$, the sphere spectrum.

Let $A = PX$ for a positive cofibrant symmetric spectrum X. We claim that to show ψ_A is a weak equivalence it suffices to show that ψ is a weak equivalence for each symmetric power $X^{(i)}/\Sigma_i$. $\mathrm{Hom}(\mathbb{I}, (\mathbb{I}\wedge -)^f)$ does not necessarily commute with coproducts, but it does commute up to weak equivalence with homotopy coproducts because it is naturally isomorphic to the identity on $\mathrm{Ho}(Sp^{\Sigma})$. Here the coproduct is a homotopy coproduct because it is created levelwise and each level of each summand is cofibrant. This is our general strategy; we follow the outline of the proof of [14, Theorem 0.7] but there the composite of the adjoints commutes with colimits on the nose and here it may only commute up to weak equivalence with homotopy colimits. But each of the colimits we must consider is in fact a homotopy colimit of symmetric spectra because such homotopy colimits can be computed levelwise [23, 2.2.1].

Now consider each summand. Since $\mathbb{I} \wedge -$ is a left Quillen adjoint on the spectrum level it commutes with colimits and smash products with spaces, so $\mathbb{I} \wedge -$ applied to Φ_i of X in Sp^Σ is isomorphic to the weak equivalence Φ_i of the cofibrant object $\mathbb{I} \wedge X$ in $\mathcal{C}$. Thus it is sufficient to show that ψ is a weak equivalence for $E\Sigma_{i+} \wedge_{\Sigma_i} X^{(i)}$ because Φ_i in Sp^Σ and $\mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge \Phi_i)^f)$ are weak equivalences by [14, 15.5] and the second hypothesis on $\mathcal{C}$. The extended power is the homotopy colimit of the Σ_i action on $X^{(i)}$. So since ψ is a weak equivalence for the positive cofibrant object $X^{(i)}$ and $\mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge -)^f)$ commutes with homotopy colimits, we conclude that ψ is a weak equivalence on the extended power.

Following [14, 15.9], we proceed by building cofibrant $\mathbb{S}$-algebras using modified generating cofibrations. Let $\Delta[n]$ denote the simplicial n-simplex and $\dot\Delta[n]$ its simplicial boundary. Let $B_*(K_+, K_+, K_+)$ be the simplicial bar construction which is the bisimplicial set with s,t-simplices $(K_s \times (\Delta[1])_t)_+$. Its geometric realization is isomorphic to $K_+ \wedge \Delta[1]_+$. The inclusion $i_1\colon \Delta[0]_+ \longrightarrow \Delta[1]_+$ induces an inclusion of the horizontally constant bisimplicial set $c_*(K_+)$ into $B_*(K_+, K_+, K_+)$. Define $B_*(K_+, K_+, S^0)$ as the pushout of this inclusion over the map $K_+ \longrightarrow S^0$. The geometric realization of the composite gives a map $K_+ \longrightarrow B(K_+, K_+, S^0)$ with the geometric realization $B(K_+, K_+, S^0)$ isomorphic to the unreduced cone $(CK)_+$. We can use these composite maps with $K = \dot\Delta[r]$ instead of the simplicially homotopic maps $\dot\Delta[r]_+ \longrightarrow \Delta[r]_+$ to construct generating cofibrations. The model category of commutative symmetric ring spectra is cofibrantly generated with $PF^+I = \{P(F_n\dot\Delta[k]_+) \longrightarrow P(F_n(C\dot\Delta[k])_+)\}_{k\geq 0, n>0}$ a set of generating cofibrations.

So we need to show that ψ_A is a weak equivalence when A is a PF^+I-cell complex [8, 2.1.18]. We have shown ψ_A is a weak equivalence when A is built in one stage. We next consider ψ_A for A constructed from PF^+I by finitely many pushouts. Assume the result for those $\mathbb{S}$-algebras built in n stages, and consider $A = A_n \wedge_{PX} PY$ with A_n built in n stages and $X \longrightarrow Y$ a coproduct of maps in PF^+I. Since F_n commutes with colimits and smash products with spaces, it commutes with geometric realization and the bar construction above. If $X = \vee_i F_{n_i}\dot\Delta[r_i]_+$ and $T = \vee_i F_{n_i} S^0$, then $Y \cong B(X, X, T)$, the geometric realization of the simplicial symmetric spectrum with q-simplices the coproduct of $q+1$ copies of X and one copy of T. Statements analogous to [14, 5.1] and [6, VII.2.10, VII.3.3] show that the category of commutative $\mathbb{S}$-algebras is tensored over simplicial sets and the underlying symmetric spectrum of the geometric realization of a simplicial commutative $\mathbb{S}$-algebra is isomorphic to the geometric realization of the underlying simplicial symmetric spectrum. Since P commutes with colimits and converts smash products with spaces to tensors with spaces, P commutes with geometric realizations. Hence,

$$A \cong A_n \wedge_{PX} PY \cong A_n \wedge_{PX} B(PX, PX, PT) \cong B(A_n, PX, PT).$$

Since tensors with simplicial sets and colimits in Sp^Σ are levelwise, the geometric realization is constructed on each level. But the geometric realization of a bisimplicial set is weakly equivalent to the homotopy colimit by [2, XII.4.3]. So the geometric realization $B(A_n, PX, PT)$ is weakly equivalent to the homotopy colimit and $\mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge -)^f)$ commutes with the homotopy colimit. So it is enough to show that ψ is a weak equivalence on each simplicial level of $B(A_n, PX, PT)$. The q-simplices here are given by $A_n \wedge (PX)^{(q)} \wedge PT \cong A_n \wedge P(X \vee \cdots \vee X \vee T)$. These q-simplices can be constructed in n stages, so by induction ψ is a weak equivalence here, as required.

Finally, we must consider a filtered colimit of these commutative $\mathbb{S}$-algebras built in finitely many stages. Filtered colimits of commutative $\mathbb{S}$-algebras are created on the underlying symmetric spectra which in turn are created on each level. Since the maps in question here are constructed from PF^+I, they are injections. So the colimit is over level injections between level cofibrant objects and it is weakly equivalent to the homotopy colimit. By induction we have shown that ψ is a weak equivalence at each spot in the colimit and $\mathrm{Hom}(\mathbb{I}, (\mathbb{I} \wedge -)^f)$ commutes

with homotopy colimits, so ψ is a weak equivalence on the colimit as well. So we conclude that ψ_A is a weak equivalence for any cofibrant commutative symmetric ring spectrum A (i.e. any retract of a PF^+I-cell complex).

For the second statement we consider cofibrant commutative R-algebras A for R a cofibrant commutative symmetric ring spectrum. Since R is cofibrant the unit map $\mathbb{S} \longrightarrow R$ is a cofibration of commutative symmetric ring spectra. Since A is cofibrant as a commutative R-algebra, the unit map $R \longrightarrow A$ and hence the composite $\mathbb{S} \longrightarrow A$ is also a cofibration. Hence, by the above, ψ_A is a weak equivalence. □

6. Non-simplicial case

In this section we consider the case when the given stable, monoidal model category $\mathcal{C}$ is not simplicial. We produce a Quillen adjoint pair from Sp^Σ to $\mathcal{C}$ whose derived functors are monoidal. This can be used for example to produce a Quillen adjoint pair from Sp^Σ to $\mathbb{Z}$-graded chain complexes. Since $\mathcal{C}$ is not simplicial, we need a new definition for a desuspension of the unit.

Definition 6.1. A *cylinder object* for a cofibrant object X is an object $X \times I$ with a factorization of the fold map $X \amalg X \xrightarrow{i} X \times I \xrightarrow{p} X$ such that i is a cofibration and p is a weak equivalence. A model for the *suspension*, ΣX, is the cofiber of $X \amalg X \xrightarrow{i} X \times I$ for some cylinder $X \times I$ [16, Chapter I §1, 2]. A *good desuspension* of the unit is a cofibrant object $\mathbb{I}_c^{-1}$ with a weak equivalence $\eta\colon \Sigma\mathbb{I}_c^{-1} \longrightarrow \mathbb{I}$ for some model of the suspension. A *stable monoidal model category* is a monoidal model category which is stable and has a good desuspension of the unit.

Theorem 6.2. *Let $\mathcal{C}$ be a stable monoidal model category. Then there is a Quillen adjoint pair from the positive stable model structure on Sp^Σ to $\mathcal{C}$, again denoted by $\mathbb{I} \wedge -$ and* $\mathrm{Hom}(\mathbb{I}, -)$*, such that the total left derived functor $\mathbb{I} \wedge^L -$ is strong monoidal. Moreover,* $\mathrm{Hom}(\mathbb{I}, -)$ *is lax monoidal, $\mathbb{I} \wedge \mathbb{S} \cong \mathbb{I}$, and $\mathbb{I} \wedge Q\mathbb{S} \longrightarrow \mathbb{I} \wedge \mathbb{S}$ is a weak equivalence.*

As with the cofibrant desuspension, the existence of a functor with the properties listed for $\mathbb{I} \wedge -$ implies the existence of a good desuspension. Under additional conditions on the unit this monoidal functor induces a monoidal equivalence of the homotopy category of $\mathcal{C}$ and the homotopy category of symmetric spectra. The proof of the following statement is similar to Theorem 4.8.

Theorem 6.3. *Let $\mathcal{C}$ be a stable monoidal model category. The following conditions are equivalent:*

1. *There is a π_*^s-linear, triangulated equivalence between the homotopy category of $\mathcal{C}$ and the homotopy category of Sp^Σ which takes the unit $\mathbb{I}$ of the monoidal product to the unit $\mathbb{S}$.*
2. *$\mathbb{I}$, is a small weak generator and $[\mathbb{I}, \mathbb{I}]_*^{\mathrm{Ho}(\mathcal{C})}$ is freely generated as a π_*^s-module by $id_{\mathbb{I}}$.*
3. *$\mathcal{C}$ and Sp^Σ are Quillen equivalent via functors whose derived functors are strong monoidal.*
4. *There is a π_*^s-linear, triangulated, monoidal equivalence between* $\mathrm{Ho}(\mathcal{C})$ *and* $\mathrm{Ho}(Sp^\Sigma)$.

To construct the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ we use *cosimplicial resolutions* since $\mathcal{C}$ is not simplicial. These were first used in [5] to construct function complexes on homotopy categories, but in [4] this theory has been extended to provide function complexes on model categories. Our main reference here is [8, Chapter 5], see also [21].

Given a cosimplicial object $X^\cdot$ in $\mathcal{C}^\Delta$ and a pointed simplicial set K denote the coend [12, Chapter IX §6] in $\mathcal{C}$ by $X^\cdot \otimes_\Delta K$ see also [8, Chapter 5 §7]. Define $X^\cdot \otimes K$ by $(X^\cdot \otimes K)^n = X^\cdot \otimes_\Delta (K \wedge \Delta[n]_+)$. Notice $X^\cdot \otimes K$ and $X^\cdot \otimes_\Delta K$ are objects in different categories ($\mathcal{C}^\Delta$ and $\mathcal{C}$ respectively.) Set $S^m = (S^1)^m$ and denote $X^\cdot \otimes S^m$ by $\Sigma^m(X^\cdot)$. If $X^\cdot$ is a cosimplicial object

and Y is an object of $\mathcal{C}$ then $\mathcal{C}(X^{\cdot}, Y)$ is a simplicial set with degree n the set of $\mathcal{C}$-morphisms $\mathcal{C}(X^n, Y)$. There is an adjunction isomorphism $\mathcal{C}(X^{\cdot} \otimes K, Y) \cong \mathrm{map}(K, \mathcal{C}(X^{\cdot}, Y))$. This shows that $X^{\cdot} \otimes (K \wedge L) \cong (X^{\cdot} \otimes K) \otimes L$ since they both represent the same functor. In particular, $\Sigma^m(X^{\cdot})$ is the mth iterated suspension of $X^{\cdot}$.

We consider the Reedy model category on $\mathcal{C}^{\Delta}$, the cosimplicial objects on $\mathcal{C}$ [17], [8, Theorem 5.2.5]. An object $A^{\cdot}$ is *Reedy cofibrant* if the map $A^{\cdot} \otimes \partial\Delta[k]_+ \longrightarrow A^{\cdot} \otimes \Delta[k]_+ \cong A^k$ is a cofibration for each k. A *cosimplicial resolution* is then a Reedy cofibrant object of $\mathcal{C}^{\Delta}$ such that each of the codegeneracy and coface maps are weak equivalences. That is, cosimplicial resolutions are the Reedy cofibrant, homotopically constant objects. A cosimplicial resolution A is called a cosimplicial frame of the cofibrant object A^0 in [8, Chapter 5].

The category $\mathcal{C}^{\Delta}$ has a symmetric monoidal product, defined on each level by the symmetric monoidal product on $\mathcal{C}$. That is, $(A^{\cdot} \wedge B^{\cdot})^n \cong A^n \wedge B^n$. The following proposition collects several useful properties of the proceeding constructions. These properties follow from [7, Theorem 16.4.2, Proposition 16.11.1] and [8, Proposition 5.7.1 and 5.7.2].

Proposition 6.4. *Let $A^{\cdot}$ and $B^{\cdot}$ be cosimplicial resolutions in $\mathcal{C}^{\Delta}$ where $\mathcal{C}$ is a monoidal model category such that $\wedge$ commutes with colimits.*

1. *$A^{\cdot} \wedge B^{\cdot}$ is a cosimplicial resolution.*
2. *$\Sigma A^{\cdot}$ is a cosimplicial resolution.*
3. *There is a natural level equivalence $\Sigma(A^{\cdot} \wedge B^{\cdot}) \longrightarrow (\Sigma A^{\cdot}) \wedge B^{\cdot}$.*

Proof. For part 1, note that $A^{\cdot} \wedge B^{\cdot}$ is homotopically constant since the smash product of two maps which are each weak equivalences between cofibrant objects is a weak equivalence. The monoidal product also preserves Reedy cofibrant objects. By [7, Proposition 16.11.1], since $\wedge\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ preserves cofibrations and $\wedge$ commutes with colimits, the prolongation $\wedge\colon \mathcal{C}^{\Delta} \times \mathcal{C}^{\Delta} \longrightarrow \mathcal{C}^{\Delta}$ also preserves cofibrations. This uses [7, Theorem 16.4.2] to recognize that the Reedy model category on $\mathcal{C}^{\Delta} \times \mathcal{C}^{\Delta}$ agrees with the Reedy model category on $(\mathcal{C} \times \mathcal{C})^{\Delta}$.

For part 2, the map $\Sigma A^{\cdot} \otimes (\partial\Delta[k]_+ \longrightarrow \Delta[k]_+)$ is isomorphic to the map $A^{\cdot} \otimes (S^1 \wedge \partial\Delta[k]_+ \longrightarrow S^1 \wedge \Delta[k]_+)$. By [8, Proposition 5.7.1], if $A^{\cdot}$ is Reedy cofibrant then this map is a cofibration. So $\Sigma A^{\cdot}$ is Reedy cofibrant. Since each map $S^1 \wedge \Delta[n]_+ \longrightarrow S^1 \wedge \Delta[n+1]_+$ is a trivial cofibration, the coface maps of $\Sigma A^{\cdot}$ are trivial cofibrations [8, Proposition 5.7.2]. Since $s^i d^i = \mathrm{id}$ the codegeneracy maps are also weak equivalences.

For part 3, the coend defining $\Sigma(A^{\cdot} \wedge B^{\cdot})^m$ is a colimit of copies of $(A^k \wedge B^k)$ indexed by the non-base point k-simplices of $S^1 \wedge \Delta[m]_+$. Use the map $k \longrightarrow m$ in Δ determined by the k-simplex of $\Delta[m]_+$ to induce a map $B^k \longrightarrow B^m$. These maps are all compatible and define a map $\Sigma(A^{\cdot} \wedge B^{\cdot})^m \longrightarrow \Sigma(A^{\cdot} \wedge B^m)^m \cong \Sigma(A^{\cdot})^m \wedge B^m \cong (\Sigma(A^{\cdot}) \wedge B^{\cdot})^m$. Here $(A^{\cdot} \wedge B^m)^k \cong A^k \wedge B^m$ and we have used the fact that $\wedge$ commutes with colimits. Since parts 1 and 2 show that this is a map between cosimplicial frames, it is a level equivalence if degree zero is a weak equivalence. The map $A^{\cdot} \wedge B^{\cdot} \longrightarrow A^{\cdot} \wedge B^0$ is a level equivalence between Reedy cofibrant objects by part 1 and the monoidal model structure on $\mathcal{C}$. Hence $(A^{\cdot} \wedge B^{\cdot}) \otimes_{\Delta} S^1 \longrightarrow (A^{\cdot} \wedge B^0) \otimes_{\Delta} S^1 \cong (A^{\cdot} \otimes_{\Delta} S^1) \wedge B^0$ is a weak equivalence by [8, Proposition 5.7.1]. □

Proof of Theorem 6.2. To define the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ we consider cosimplicial resolutions related to $\mathbb{I}$. First, let $\omega^0\mathbb{I}$ be the constant cosimplicial object on $\mathbb{I}$. Since $\mathbb{I}$ is not necessarily cofibrant, $\omega^0\mathbb{I}$ is not necessarily a cosimplicial resolution. Since $\mathcal{C}$ has a good desuspension of the unit, one can build a cosimplicial resolution $\omega^1\mathbb{I}$ of the cofibrant object $\mathbb{I}_c^{-1}$. Define $(\omega^1\mathbb{I})^0 = \mathbb{I}_c^{-1}$ and $(\omega^1\mathbb{I})^1 = \mathbb{I}_c^{-1} \times I$. Define the coface maps as the two inclusions $X \longrightarrow X \amalg X \xrightarrow{i} X \times I$ and define the codegeneracy map as the map $X \times I \xrightarrow{p} X$. Using the factorization properties in $\mathcal{C}$ one can inductively define the higher levels of $\omega^1\mathbb{I}$, see the proof of [8, Theorem 5.1.3]. Since $\Sigma(X^{\cdot})$ is the cofiber of $X^{\cdot} \otimes (S^0 \vee S^0) \longrightarrow X^{\cdot} \otimes \Delta[1]_+$, $\Sigma(\omega^1\mathbb{I})^0$ is the cofiber of i, that

is, a model for the suspension of $\mathbb{I}_c^{-1}$. Then the weak equivalence $\eta\colon\ \Sigma\mathbb{I}_c^{-1} \longrightarrow \mathbb{I}$ extends to a level equivalence $\eta^{\cdot}\colon\ \Sigma\omega^1\mathbb{I} \longrightarrow \omega^0\mathbb{I}$. Define $\omega^n\mathbb{I} = (\omega^1\mathbb{I})^{\wedge n}$ for $n > 0$. Define the right adjoint $\mathrm{Hom}(\mathbb{I}, -)$ in level n to be $\mathcal{C}(\omega^n\mathbb{I}, -)$. The symmetric group on n letters permutes the factors of $\omega^n\mathbb{I}$. The structure maps are induced by the map η. Proposition 6.4 part 3 provides a level equivalence $\phi\colon\ \Sigma^m(A^{\wedge m} \wedge B) \longrightarrow (\Sigma A)^{\wedge m} \wedge B$. The isomorphism of $\Sigma^m(X^{\cdot})$ with the m-fold iterated suspension of $X^{\cdot}$ induces a $\Sigma_m \times \Sigma_n$-equivariant level equivalence where Σ_m acts trivially on the target:

$$\Sigma^m(\omega^{m+n}\mathbb{I}) \xrightarrow{\phi} (\Sigma\omega^1\mathbb{I})^{\wedge m} \wedge \omega^n\mathbb{I} \xrightarrow{(\eta^{\wedge m})\wedge \mathrm{id}} (\omega^0\mathbb{I})^{\wedge m} \wedge \omega^n\mathbb{I} \cong \omega^n\mathbb{I}.$$

Applying $\mathcal{C}(-, Z)$ to the displayed composition and taking adjoints gives the $\Sigma_m \times \Sigma_n$-equivariant structure map

$$S^m \wedge \mathcal{C}(\omega^n\mathbb{I}, Z) \longrightarrow \mathcal{C}(\omega^{m+n}\mathbb{I}, Z).$$

Let $Q\omega^0\mathbb{I}$ denote the constant cosimplicial object on $Q\mathbb{I}$, the chosen cofibrant replacement of $\mathbb{I}$. Then since degree zero of η factors through $Q\mathbb{I}$, η factors as two level equivalences $\Sigma\omega^1\mathbb{I} \longrightarrow Q\omega^0\mathbb{I} \longrightarrow \omega^0\mathbb{I}$. Hence $(\eta)^{\wedge m} \wedge \mathrm{id}_{A^{\cdot}}$ for any cosimplicial resolution $A^{\cdot}$ is a level equivalence by the monoidal model structure on $\mathcal{C}$. Since $\omega^n\mathbb{I}$ for $n > 0$ is a cosimplicial resolution by Proposition 6.4 part 1, each map $\Sigma^m(\omega^{m+n}\mathbb{I}) \longrightarrow \omega^n\mathbb{I})$ with $n > 0$ is a weak equivalence. By the pointed version of [8, Corollary 5.4.4], $\mathcal{C}(A^{\cdot}, -)$ preserves fibrations and trivial fibrations when $A^{\cdot}$ is a cosimplicial resolution and for Z fibrant $\mathcal{C}(-, Z)$ takes level equivalences between cosimplicial resolution to weak equivalences. So $\mathrm{Hom}(\mathbb{I}, -)$ takes fibrant objects to positive Ω-spectra and (trivial) fibrations to positive level (trivial) fibrations. Thus $\mathrm{Hom}(\mathbb{I}, -)$ is a right Quillen functor by [3, Corollary A.2] since positive stable fibrations between positive Ω-spectra are positive level fibrations by Proposition 4.3. The left adjoint, $\mathbb{I} \wedge -$ is formed as in the simplicial case except here the tensors of cosimplicial resolutions with simplicial sets are given by coends.

To show that the total left derived functor $\mathbb{I} \wedge^L -$ is strong monoidal we first show that $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal. For the unit map take the non-base point of S^0 to the identity map in simplicial degree zero of $\mathrm{Hom}(\mathbb{I}, \mathbb{I})^0$. The monoidal product on $\mathcal{C}$ induces a natural map $\mathcal{C}(\omega^m\mathbb{I}, A) \wedge \mathcal{C}(\omega^n\mathbb{I}, B) \longrightarrow \mathcal{C}(\omega^{m+n}\mathbb{I}, A \wedge B)$. Assembling these levels produces a natural map $\mathrm{Hom}(\mathbb{I}, A) \wedge \mathrm{Hom}(\mathbb{I}, B) \longrightarrow \mathrm{Hom}(\mathbb{I}, A \wedge B)$. Hence, $\mathrm{Hom}(\mathbb{I}, -)$ is lax monoidal. So its left adjoint $\mathbb{I} \wedge -$ is lax comonoidal. Also, $\mathbb{I} \wedge F'_n X \cong (\omega^n\mathbb{I} \otimes_{\Sigma_n} X)^0$ because they represent the same functor in $\mathcal{C}$. So $\mathbb{I} \wedge -$ takes the cofibrant replacement $Q\mathbb{S} \cong F'_1 S^1 \longrightarrow F_0 S^0 \cong \mathbb{S}$ to the weak equivalence $\eta\colon\ (\omega^1\mathbb{I} \otimes S^1)^0 \longrightarrow \mathbb{I}$. The comonoidal structure on $\mathbb{I} \wedge -$ induces a natural transformation $\mathbb{I} \wedge^L (A \wedge^L B) \longrightarrow (\mathbb{I} \wedge^L A) \wedge^L (\mathbb{I} \wedge^L B)$ where $\mathbb{I} \wedge^L -$ is the total left derived functor of $\mathbb{I} \wedge -$. Since $\mathbb{I} \wedge^L \mathbb{S} \cong \mathbb{I}$, this map is an isomorphism for $A = \mathbb{S}$ and any B. For fixed B both the source and target are exact functors in A which commute with coproducts, so the objects A where this transformation is an isomorphism form a localizing subcategory which contains the generator $\mathbb{S}$. Hence this transformation is an isomorphism for all A and B. So $\mathbb{I} \wedge^L -$ is strong monoidal. $\square$

Remark 6.5. Let $\mathcal{C}$ be a monoidal model category with a Quillen adjoint pair between $\mathcal{C}$ and the positive stable model category on Sp^{Σ} with left adjoint $L\colon\ Sp^{\Sigma} \longrightarrow \mathcal{C}$. If $L(Q\mathbb{S}) \longrightarrow L(\mathbb{S})$ is a weak equivalence and $L(\mathbb{S}) \cong \mathbb{I}$ then $\mathcal{C}$ has a good desuspension of the unit. Define $\mathbb{I}_c^{-1} = L(F_1 S^0)$, with cylinder $L(F_1\Delta[1]_+)$ and model of the suspension $L(F_1 S^1)$. These definitions have all the necessary properties since L preserves positive cofibrations and weak equivalences between positive cofibrant objects. Define $\eta\colon\ L(F_1 S^1) \longrightarrow L(F_0 S^0)$ as the adjoint of the identity map on level one. In fact, the cosimplicial resolution $\omega^1\mathbb{I}$ can be defined by $(\omega^1\mathbb{I})^n = L(F_1\Delta[n]_+)$.

References

[1] J. M. Boardman, *Stable homotopy theory*, Mimeographed notes, Johns Hopkins University, 1969–70.

[2] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag 1972.
[3] D. Dugger, *Replacing model categories with simplicial ones*, Trans. Amer. Math. Soc., to appear.
[4] W. G. Dwyer, P. S. Hirschhorn, and D. M. Kan, *Model categories and general abstract homotopy theory*, Preprint 1997, `http://www-math.mit.edu/~psh/`.
[5] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), 427–440.
[6] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, **47**, American Mathematical Society, Providence, RI, 1997, xii+249 pp.
[7] P. S. Hirschhorn, *Localization, cellularization, and homotopy colimits*, Preprint 1999, `http://www-math.mit.edu/~psh/`.
[8] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, **63**, American Mathematical Society, Providence, RI, 1999, xii+209 pp.
[9] M. Hovey, J. H. Palmieri, and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610.
[10] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.
[11] M. Lydakis, *Simplicial functors and stable homotopy theory*, Preprint, Universität Bielefeld, 1998.
[12] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Math. **5**, Springer, New York-Berlin, 1971, ix+262 pp.
[13] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Preprint 2000.
[14] M. A. Mandell, J. P. May, S. Schwede and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc., to appear, `http://hopf.math.purdue.edu/`.
[15] H. R. Margolis, *Spectra and the Steenrod algebra. Modules over the Steenrod algebra and the stable homotopy category*, North-Holland Mathematical Library **29**, North-Holland Publishing Co., Amsterdam-New York, 1983, xix+489 pp.
[16] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Math. **43**, Springer-Verlag, 1967.
[17] C. L. Reedy, *Homotopy theory of model categories*, Preprint 1974, `http://www-math.mit.edu/~psh/`.
[18] C. Rezk, S. Schwede and B. Shipley, *Simplicial structures on model categories and functors*, to appear in Amer. J. of Math., `http://www.math.purdue.edu/~bshipley/`.
[19] S. Schwede, *S-modules and symmetric spectra*, to appear in Math. Ann., `http://hopf.math.purdue.edu/`.
[20] S. Schwede and B. Shipley, *Algebras and modules in monoidal model categories*, Proc. of the London Math. Soc. **80** (2000), 491-511.
[21] S. Schwede and B. Shipley, *A uniqueness theorem for stable homotopy theory*, Preprint 1999, `http://www.math.purdue.edu/~bshipley/`.
[22] S. Schwede and B. Shipley, *Classification of stable model categories*, Preprint 1999, `http://www.math.purdue.edu/~bshipley/`.
[23] B. Shipley, *Symmetric spectra and topological Hochschild homology*, K-Theory **19** (2000), 155-183.
[24] E. H. Spanier and J. H. C. Whitehead, *A first approximation to homotopy theory*, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 655–660.
[25] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis, xii+253 pp.
[26] R. Vogt, *Boardman's stable homotopy category*, Lecture Notes Series, **21**, Matematisk Institut, Aarhus Universitet (1970).

Department of Mathematics, University of Chicago, Chicago, IL 60637 and,

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address: `bshipley@math.uchicago.edu, bshipley@math.purdue.edu`